On a generalization of Chen's iterated integrals

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July 22, 2008

Abstract

Chen's iterated integrals may be generalized by interpolation of functions of the positive integer number of times which particular forms are iterated in integrals along specific paths, to certain complex values. These generalized iterated integrals satisfy both an additive and a (non-classical) multiplicative iterative property, in addition to a comultiplication formula. This theory is developed in the first part of the paper, after which various applications are discussed, including the expression of certain zeta functions as complex iterated integrals (from which an obstruction to the existence of a contour integration proof of the functional equation for the Dedekind zeta function emerges); an elegant reformulation of a result of Gel'fand and Shilov in the theory of distributions which gives a way of thinking about complex iterated derivatives; and a direct topological proof of the monodromy of polylogarithms. [1]

0 Introduction

The iterated integrals of K.-T. Chen arise in arithmetic situations, a famous example of which is the occurrence of the polyzeta values (also called multiple zeta values) as periods relating two distinct rational structures on the mixed Hodge structure which comprises the Hodge realization of the motivic fundamental group of $\mathbb{P}^1\backslash\{0,1,\infty\}$ with tangential base-point $\overrightarrow{01}$. In this paper, it is shown that more general objects, including the polyzeta functions themselves, may be viewed as iterated integrals of a sort generalizing the notion introduced by Chen, and the eventual hope is that such objects could thereby also acquire further arithmetic significance.

A very general formulation of these iterated integrals is presented in the first section of the paper, in which it is shown that formal generalizations of the antipode and product formulas satisfied by Chen's integrals may be ideated and then exploited to define complex iterated integrals along paths for which it is possible to prove a certain iterative property. In particular, whenever the relevant integrals converge, and the necessary iterative property may be established, then for differential 1-forms α and β on some differential manifold M on which γ is a piece-wise smooth path, we define

$$\int_\gamma \alpha\beta^{s-1} := (-1)^s \int_1^0 \frac{1}{\Gamma(s)} \left(\int_1^z \gamma^*\beta\right)^{s-1} \gamma^*\alpha(z)$$

where z is a parameter on $[0,1]$ for the pullback of α under γ, and s is some complex number.

This definition admits of the proof of a comultiplication formula extending that on the usual iterated integrals, given by

$$\int_{\gamma\delta} \alpha\beta^s = \int_\delta \alpha\beta^s + \sum_{n=0}^{\infty} \int_\gamma \alpha\beta^n \int_\delta \beta^{s-n}$$

[1] *Mathematics Subject Classification*: Primary 11G55; Secondary 11M99, 33E20. Key-words: polyzeta functions, Riemann zeta function, Dedekind zeta function, multiple zeta values, iterated integrals, iterated derivatives, monodromy of polylogarithms.

where γ and δ are paths which may be concatenated, and

$$\int_{\delta} \beta^{s-n} \tag{1}$$

is interpreted as

$$\binom{s}{n} \cdot \frac{\int_{\delta} \beta^{s}}{\int_{\delta} \beta^{n}}$$

for those n for which (1) does not converge. The formula is subject to certain technical conditions which ensure convergence of the sum.

An interesting application of this formula is to give a direct proof of the monodromy of the polylogarithm functions, which is preferable to the classical proof entailing use of Jonquière's formula. This is discussed in the last section of the paper.

Before getting to this, we develop the theory in a most interesting example, namely that of $M = \mathbb{P}^1\backslash\{0,1,\infty\}$. In this case, taking $\beta = \dfrac{dz}{z}$, the coincidence of our notion along paths $\gamma = [t,1]$ with the classical fractional integral is shown, from which it emerges that the necessary iterative property was known classically. We also demonstrate that this iterative property characterizes the complex iterated integrals in the case of $\beta = \dfrac{dz}{1-z}$.

Furthermore, we use the flexible definition of the preceding section to illustrate that complex iterated integration works over many different paths in $\mathbb{P}^1\backslash\{0,1,\infty\}$.

Beyond this, it is also possible to extend the formalism to multiple versions of complex iterated integrals, and we carry this out with a view towards expressing the polyzeta *functions* as iterated integrals.

Also, a non-classical multiplicative iterative property arises. This has an amusing consequence for the Riemann zeta function, which is shown to admit a complex iterated integral expression

$$\zeta(s) = \int_{[0,1]} \frac{dz}{1-z} \left(\frac{dz}{z}\right)^{s-1}$$

that corresponds to Abel's integral

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^s}{e^x - 1} \frac{dx}{x} \,.$$

Using the multiplicative iterativity, we arrive at a family of expressions for $\zeta(s)$ indexed by positive integers k, in which the integral corresponding to $k = 2$ is nothing other than the theta function integral which is the basis of the Fourier analysis proof of the functional equation for $\zeta(s)$.

It is also noteworthy that $\zeta(s)$ may be regarded as an integral transform of the rational function $\dfrac{z}{1-z}$, in keeping with the general philosophy that zeta functions should be rational. A similar statement holds for the Dirichlet L-functions. However, an interesting difference is that while $\dfrac{z}{1-z}$ has a pole at $z = 1$, the rational function corresponding to $L(s,\chi)$ for non-trivial Dirichlet character χ is non-singular at $z = 1$. On the other hand, $\zeta(s)$ is singular at $s = 1$, while $L(s,\chi)$ has no pole there. Although there is *a priori* no connection between the z and s coordinates, this correspondence turns out to hold quite generally. In fact, we can prove that if $F(z)$ satisfies a suitable boundedness condition (guaranteeing convergence of the relevant integral), and is meromorphic in a

neighborhood of $z=1$, then

$$L(F)(s) := \int_{[0,1]} F(z)\frac{dz}{z}\left(\frac{dz}{z}\right)^{s-1}$$

has a pole at $s=1$ if and only if $\frac{F(z)}{z}$ has a pole of non-zero residue at $z=1$; and should $L(F)(s)$ have a pole at $s=1$, the residue is a sum of coefficients of the LAURENT series expansion of F about $z=1$. A consequence of this theorem is that the function $F_K(z)$ associated by means of a complex iterated integral expression to the DEDEKIND zeta function of any field for which the residue of the pole at $s=1$ is irrational (which is expected to hold for all number fields other than $\mathbb{Q}$), is not meromorphic and hence not rational. Consequently, by a result due to FATOU, we can say that the function $F_K(z)$ is not even algebraic. Moreover, by a theorem of PETERSSON, it follows that this function is not analytically continuable beyond the unit disc. This is an obstruction to the existence of a proof of the analytic continuation and functional equation for the DEDEKIND zeta function using the contour integral approach of RIEMANN'S first proof of the corresponding facts for $\zeta(s)$.

The formalism has proven useful in explaining a well-known result of GELFON'D and SHILOV to the effect that the generalized function

$$\frac{x_+^{s-1}}{\Gamma(s)}$$

admits an analytic continuation which at the negative integer $-n$ is the same as the n th derivative DIRAC measure $\delta^{(n)}$ - i.e. the value of a test function $\phi(x)$ against the generalized function

$$\frac{x_+^{s-1}}{\Gamma(s)}|_{s=-n}$$

over the reals is given by

$$\left(-\frac{dx}{x}\right)^n \phi(x)|_{x=0}.$$

In terms of complex iterated integrals (via a change of variables) this can be elegantly reformulated as follows:

If $F(z)=\sum_{n=1}^{\infty} a_n z^n$ is holomorphic on the unit disk centered at $z=0$, has $a_n = O(n^r)$ for some $r>0$, and is also analytic in some neighborhood of $z=1$, then with notation as before, $L(F)(s)$ admits an analytic continuation with poles at most at $s=1,2,\ldots,r+1$, which has values at negative integers $-k$ given by

$$\left(z\frac{d}{dz}\right)^k F(z)|_{z=1}.$$

This result shows that we should think about the differential operator

$$\left(z\frac{d}{dz}\right)^t (\cdot)|_{z=1}$$

as the analytic continuation of

$$\int_{[0,1]} (\cdot)\left(\frac{dz}{z}\right)^s$$

to $s=-t$.

RIEMANN'S integral expression for the analytic continuation of $\zeta(s)$ may be modified to give a proof.

Altering this proof in turn, the remarkable fact emerges that for any $w \in (0,1)$,

$$\int_{[w,1]} F(z)\frac{dz}{z}\left(\frac{dz}{z}\right)^{s-1}$$

has the same analytic continuation to negative integers as does $L(F)(s)$. This fact may also be proven using the coproduct formula.

The author would like to thank Professor Minhyong Kim for his continued encouragement and valuable suggestions.

1 Iterated integrals along paths on general complex manifolds

Suppose throughout that α and β are holomorphic 1-forms on a complex manifold M and γ is a piecewise smooth path in M. The task at hand is to define

$$\int_\gamma \alpha\beta^{s-1}$$

as an iterated integral, for suitable *complex* s . For this to be well-defined, we would necessarily have to know that

$$\int_\gamma \alpha\beta^{s-1} = \int_\gamma (\alpha\beta^v)\beta^{w-1} \tag{2}$$

always holds for suitable v with $w = s - v$. In the absence of a general proof of this equality, we instead restrict ourselves to defining the complex iterated integrals only in those cases where a sufficient condition for this iterativity property (2) to hold may be established:

Definition 0 *The iterativity condition for the triple* (M, β, γ) *with respect to the pair* $(v, w) \in \mathbb{C}^2$ *is said to hold whenever the identity*

$$\frac{1}{\Gamma(v+w)}\left(\int_t^1 \gamma^*\beta\right)^{v+w-1} = \frac{1}{\Gamma(v)\Gamma(w)}\int_t^1 \left(\int_t^z \gamma^*\beta\right)^{v-1}\left(\int_z^1 \gamma^*\beta\right)^{w-1}\gamma^*\beta \tag{3}$$

is valid, where $\gamma^*\beta$ *is the pullback to* $[0,1]$ *;* z *and* t *are parameters on the interval; and all of the integrals which appear converge.*

Another property which our generalization should have is adherence to some kind of shuffle product generalizing the product on the Hopf algebra of Chen's iterated integrals. The suitable form of this generalized product is not obvious, but repeated application of the usual shuffle product formula and use of a simple induction argument shows that for any positive integer n ,

$$\left(\int_\gamma \beta\right)^n = n!\int_\gamma \beta^n. \tag{4}$$

Here, the n-fold integration on the right side is reduced to a single integration on the left side. For example, when $\gamma = [0,1]$ in $M = \mathbb{C}$, geometrically this equation gives a transition between integration over the n-cube $[0,1]^n$ (the integral on the left side is an n-fold product of equal integrals, which by Fubini's Theorem may be considered as a single integral over the cube) and integration over the time-ordered n-simplex

$$\{(t_1,\ldots,t_n) \in \mathbb{R}^n | 0 \le t_1 \le \ldots \le t_n \le 1\}$$

(the integral on the right side, by the definition of iterated integrals). There are $n!$ such simplices which together form the n-cube and the permutation of the t_j which shows this gives a change of variables yielding $n!$ equal integrals, the sum of which is the integral over the cube.

The n in (4) can be interpolated to other complex arguments in an essentially unique way: The gamma function is the unique function interpolating $n!$ having certain nice properties (namely it satisfies the functional equation $\Gamma(x+1) = x\Gamma(x)$, has $\Gamma(1) = 1$, and when restricted to the positive reals has convex logarithm). Moreover, raising to the nth power is uniquely interpolated to other complex values via

$$x^s = \exp(s\mathrm{Log}\, x),$$

once a choice of the logarithm has been made, say $\mathrm{Log}\, z = \log|z| + i\arg(z) + 2\pi i r$ for some $r \in \mathbb{Z}$ with $-\pi < \arg(z) < \pi$ (i.e. branch cut along the negative reals, or what is the same, Log has domain $X\backslash\mathbb{R}_{<0}$).

The reason that this fact is significant is that in defining some kind of complex power of the iterated integral - i.e. ascribing meaning to integration against some object which gives a valid interpretation of complex power of a differential form - we have to somehow bypass integrating "s number of times" for complex variable s.

Recall now that the antipode property of iterated integrals is

$$\int_\gamma \omega_0 \ldots \omega_r = (-1)^{r+1} \int_{\gamma^{-1}} \omega_r \ldots \omega_0 \tag{5}$$

where γ^{-1} is the inverse path to γ defined by $\gamma^{-1}(t) = \gamma(1-t)$. Along with the shuffle product, we exploit the obvious analogue of this antipode property to make the definition itself:

Definition 1 *Suppose that the iterativity condition holds for (M, β, γ) with respect to all pairs $(v, s-v)$ in some open subset of $\mathbb{C}^2$. Also let*

$$p(z) = -\int_1^z \gamma^*\beta$$

where z is a parameter on $[0,1]$ for $\gamma^*\alpha$. Then

$$\int_\gamma \alpha\beta^{s-1} := \int_0^1 \frac{(p(z))^{s-1}}{\Gamma(s)} \gamma^*\alpha(z)$$

for all s for which the integral on the right hand side converges.

Notice that here, by a formal use of the above-mentioned generalizations of (4) and (5), we have

$$\begin{aligned}
\int_0^1 \frac{(p(z))^{s-1}}{\Gamma(s)} \gamma^*\alpha(z) &= -\frac{1}{\Gamma(s)} \int_1^0 \left(-\int_1^z \gamma^*\beta\right)^{s-1} \gamma^*\alpha(z) \\
&= (-1)^s \int_1^0 \left(\int_{[1,z]} \gamma^*\beta^{s-1}\right) \gamma^*\alpha \ : \text{formal use of (4)} \\
&= \int_{[0,1]} (\gamma^*\alpha)(\gamma^*\beta)^{s-1} \qquad : \text{formal use of (5)} \\
&= \int_\gamma \alpha\beta^{s-1}.
\end{aligned}$$

It would be interesting to give some geometric interpretation of this definition along the lines of the above discussion involving simplices.

At any rate, the validity of (2) is now immediate from a direct computation.

We can also formalize the antipode property which is built into the definition:

Proposition 0 (Antipode Property) *For any α, β and s for which the integral is defined,*

$$\int_\gamma \alpha\beta^s = (-1)^{s+1} \int_{\gamma^{-1}} \beta^s \alpha \ .$$

Here the notation on the right side represents

$$(-1)^{s+1} \int_1^0 \left(\int_{[1,t]} [(\gamma^{-1})^* \beta]^s \right) (\gamma^{-1})^* \alpha,$$

where the integral over $[1,t]$ of the pull-back of β should again be interpreted as in the definition, writing $\beta^s = \beta \cdot \beta^{s-1}$.

1.1 The comultiplication formula

It will be convenient to introduce the following notation:

$$\int_{\gamma \to z} \beta := \int_0^z \gamma^* \beta.$$

Then we have the

Theorem 0 *Suppose that α and β are 1-forms on some manifold M and γ and δ are paths on M for which*

$$\int_\gamma \alpha\beta^s \ \ \textit{and} \ \ \int_\delta \alpha\beta^s$$

both converge. Suppose also that

$$\left| \int_{\delta^{-1}} \beta \right| > \left| \int_{\gamma^{-1} \to z} \beta \right|,$$

and that for any z and sufficiently large N,

$$\sum_{n=0}^{N} \binom{s}{n} \left(\int_{\gamma^{-1} \to z} \beta \right)^n \left(\int_{\delta^{-1}} \beta \right)^{-n}$$

is dominated by the limit as $N \to \infty$. Then

$$\int_{\gamma\delta} \alpha\beta^s = \int_\delta \alpha\beta^s + \sum_{n=0}^{\infty} \int_\gamma \alpha\beta^n \int_\delta \beta^{s-n}$$

where we interpret

$$\int_\delta \beta^{s-n} \tag{6}$$

as

$$\binom{s}{n} \cdot \frac{\int_\delta \beta^s}{\int_\delta \beta^n}$$

whenever (6) does not converge.

Proof:

$$\begin{aligned}
\int_{\gamma\delta}\alpha\beta^s &= \frac{-1}{\Gamma(s+1)}\int_{\delta^{-1}\gamma^{-1}}\left(-\int_{(\delta^{-1}\gamma^{-1})\to z}\beta\right)^s\alpha(z)\\
&= \frac{-1}{\Gamma(s+1)}\int_{\delta^{-1}}\left(-\int_{\delta^{-1}\gamma^{-1}\to z}\beta\right)^s\alpha(z)+\frac{-1}{\Gamma(s+1)}\int_{\gamma^{-1}}\left(-\int_{\delta^{-1}\gamma^{-1}\to z}\beta\right)^s\alpha(z)\\
&= \frac{-1}{\Gamma(s+1)}\int_{\delta^{-1}}\left(-\int_{\delta^{-1}\to z}\beta\right)^s\alpha(z)+\frac{-1}{\Gamma(s+1)}\int_{\gamma^{-1}}\left(-\int_{\delta^{-1}}\beta-\int_{\gamma^{-1}\to z}\beta\right)^s\alpha(z)\\
&= \int_\delta\alpha\beta^s+\frac{-1}{\Gamma(s+1)}\int_{\gamma^{-1}}\left(-\int_{\delta^{-1}}\beta\right)^s\left(\sum_{n=0}^\infty\binom{s}{n}\left(-\int_{\delta^{-1}}\beta\right)^{-n}\left(-\int_{\gamma^{-1}\to z}\beta\right)^n\right)\alpha(z)\\
&= \int_\delta\alpha\beta^s+\frac{-1}{\Gamma(s+1)}\sum_{n=0}^\infty\left(\left(-\int_{\delta^{-1}}\beta\right)^{s-n}\right)\cdot\binom{s}{n}\int_{\delta^{-1}\gamma^{-1}}\left(-\int_{\gamma^{-1}\to z}\beta\right)^n\alpha(z)\\
&= \int_\delta\alpha\beta^s+\frac{1}{\Gamma(s+1)}\sum_{n=0}^\infty\Gamma(s-n+1)\int_\delta\beta^{s-n}\cdot\binom{s}{n}n!\int_\gamma\alpha\beta^n\\
&= \int_\delta\alpha\beta^s+\sum_{n=0}^\infty\int_\gamma\alpha\beta^n\cdot\int_\delta\beta^{s-n}
\end{aligned}$$

using the binomial series. □

Of course, along with the coproduct formula, one would like some kind of product formula so as to have a Hopf algebra of complex iterated integrals. It is obvious what such a formula would have to look like, but at this point there is a difficulty in the interpretation of the meaning of certain integrals in this formula, so the resolution of this problem will have to await further work.

Notice that in the coproduct formula, one only shifts by integers. For this reason, a less general iterative property than that discussed above probably suffices (i.e. in (2) we would only need to consider pairs (s,w) where one or other of the entries is an integer).

2 Integrating on $\mathbb{P}^1\backslash\{0,1,\infty\}$

Deligne's epic work [De] establishes the fundamental group of $\mathbb{P}^1\backslash\{0,1,\infty\}$ as an interesting object of study, providing as it does a test-case for the motivic philosophy. In work of Wojtkoviak and Drinfel'd related to this study, values of the Riemann zeta function made a surprise appearance. This phenomenon is now well-understood: The polyzeta numbers (also called multiple zeta values in the literature), are periods relating two distinct rational structures on the mixed Hodge structure which comprises the Hodge realization of the motivic fundamental group of $\mathbb{P}^1\backslash\{0,1,\infty\}$ with tangential base-point $\overrightarrow{01}$. A fundamental reason for this is that each polyzeta number admits an expression as an iterated integral in the sense of Chen over the holomorphic 1-forms of $\mathbb{P}^1\backslash\{0,1,\infty\}$. By studying the complex iterated integrals defined above in the context of $\mathbb{P}^1\backslash\{0,1,\infty\}$, we are able to realize the polyzeta *functions* - along with other generalizations of the Riemann zeta function - as iterated integrals along the tangential path from 0 to 1.

Preliminary to this investigation, we explain the regularization of the logarithm at zero which will be used: Suppose that $f(z)$ is defined in some neighborhood U of zero from which the points along the

negative real axis have been deleted, and assume that for ε close to zero and for $b \in U$,

$$\int_{\varepsilon}^{b} f(z)dz = b_0 + b_1 \log \varepsilon + b_2 (\log \varepsilon)^2 + \dots .$$

Then we set

$$\int_{0}^{b} f(z)dz := b_0.$$

We shall use the notation

$$\mathrm{Rlim}_{\,\varepsilon\to 0}(b_0 + b_1 \log \varepsilon + b_2(\log \varepsilon)^2 + \dots) = b_0$$

for the regularized limit so that in particular,

$$\int_0^t \frac{dz}{z} = \mathrm{Rlim}_{\,\varepsilon\to 0^+} (\log t - \log \varepsilon) = \log t.$$

By the same token, we regularize at $z = 1$ by setting

$$\int_a^1 f(z)dz := a_0$$

where $a \in V$, some neighborhood of $z = 1$ from which the points of $[1, \infty)$ have been deleted[2], whenever

$$\int_a^{1-\delta} f(z)dz = a_0 + a_1 \log \delta + a_2 (\log \delta)^2 + \dots$$

for δ close to zero. In particular, we interpret the integral over the path from 0 to 1 as the sum of regularized limits

$$\int_0^1 f(z)dz = \mathrm{Rlim}_{\,\varepsilon\to 0} \int_{\varepsilon}^{a} f(z)dz + \mathrm{Rlim}_{\,\delta\to 0} \int_a^{1-\delta} f(z)dz$$

for any $a \in (0,1)$ where $f(z)$ is defined in some suitable open set containing $(0,1)$.

Furthermore, to ensure convergence of the integral, we place a restriction on the functions f which are being integrated.

Definition 2 *Let $k \in \mathbb{Z}_{\geq 0}$. A k-BIEBERBACH function is a function $f(z)$ which is holomorphic on the punctured unit disk $D'(0,1) := \{z \in \mathbb{C} : 0 < |z| < 1\}$ and has a LAURENT series expansion*

$$f(z) = \sum_{n \geq m} a_n z^n$$

which satisfies the following property: k is minimal for which there exist positive N_k and C_k so that

$$|a_n| \leq C_k n^k$$

whenever $n \geq N_k$. (i.e. $a_n = O(n^k)$.)
We shall say that a function is at least k-BIEBERBACH if it is l-BIEBERBACH for some $l \leq k$.

Examples:
0. Schlict functions are 1-BIEBERBACH (DE BRANGES).
1. $F_{\mathbb{Q}}(z) := \dfrac{z}{1-z}$ is 0-BIEBERBACH. Later we show that $F_{\mathbb{Q}}$ underlies the RIEMANN zeta function.

[2] To be most general and in fact allowing for V to be of the form of the punctured unit disc about $z = 1$, V should be taken to be a neighborhood of the tangential basepoint $\vec{10}$.

Lemma 0 *Suppose* $f(z)$ *is at least* k -Bieberbach. *Then*

$$\int_0^1 \frac{(-\log z)^{s-1}}{\Gamma(s)} f(z)\frac{dz}{z} = \int_{[0,1]} f(z)\frac{dz}{z}\left(\frac{dz}{z}\right)^{s-1}$$

converges for $Re(s) > k+1$.
We shall henceforth use the notation

$$\int_{[0,1]} f(z)\left(\frac{dz}{z}\right)^{s}$$

for this integral.
Proof: For $v \neq 0$ and any $c \in (0,1)$,

$$\int_0^c \frac{(\log c - \log z)^{s-1}}{\Gamma(s)} z^v \frac{dz}{z} = \frac{c^v}{v^s}$$

via the substitution $z^v = c^v u$ and use of the definition of $\Gamma(s)$; while if $v=0$, this integral regularizes as

$$\operatorname{Rlim}_{\varepsilon\to 0} -\frac{(\log c - \log t)^s}{\Gamma(s+1)}\bigg|_{\varepsilon}^{c} = \operatorname{Rlim}_{\varepsilon\to 0}\left[-\frac{(0)^s}{\Gamma(s+1)} + \frac{(\log c - \log\varepsilon)^s}{\Gamma(s+1)}\right] = \frac{(\log c)^s}{\Gamma(s+1)}$$

using analyticity of raising to the s power near $\log c \neq 0$.)

The Laurent series expression for f converges uniformly on compacta in $D'(0,1)$. Hence the order of the summation of this series and integration over subintervals $[a,b]$ with $0<a<b<1$ may be interchanged. Consequently, if

$$f(z) = \sum_{n\geq m} a_n z^n$$

on $D'(0,1)$, then for any $c \in (0,1)$,

$$\begin{aligned}
\int_0^c \frac{(\log c - \log z)^{s-1}}{\Gamma(s)} f(z)\frac{dz}{z} &= \lim_{\varepsilon\to 0}\int_\varepsilon^c \frac{(\log c - \log z)^{s-1}}{\Gamma(s)} f(z)\frac{dz}{z} \\
&= \lim_{\varepsilon\to 0}\sum_{n\geq m}\int_\varepsilon^c \frac{(\log c - \log z)^{s-1}}{\Gamma(s)} a_n z^n\frac{dz}{z} \\
&= a_0\frac{(\log c)^s}{\Gamma(s+1)} + \sum_{n\geq m; n\neq 0}\frac{a_n c^n}{n^s}.
\end{aligned}$$

Now one can take the limit as c approaches 1. This gives

$$\int_0^1 \frac{(-\log z)^{s-1}}{\Gamma(s)} f(z)\frac{dz}{z} = \sum_{n\geq m}\frac{a_n}{n^s},$$

should the sum converge. Now since $|a_n| \leq C_k n^k$ for all $n \geq N_k$, then

$$\left|\sum_{n\geq N}\frac{a_n}{n^s}\right| \leq \sum_{n\geq N}\left|\frac{a_n}{n^s}\right| \leq C_k\sum_{n\geq N}\frac{1}{n^{\sigma-k}}$$

where $Re(s)=\sigma$, which converges provided $\sigma - k > 1$. $\square$

It is not difficult to establish the iterativity condition for the path $[0,1]$ - or for that matter, for any straight line path in $\mathbb{P}^1\backslash\{0,1,\infty\}$, or the small loops about 0 or 1 respectively. In these cases, the content of the condition is nothing other than the classical equality of the beta function

$$\beta(s,w) = \int_0^1 a^{s-1}(1-a)^{w-1}da \tag{7}$$

with the quotient of gamma functions:

$$\frac{\Gamma(s)\Gamma(w)}{\Gamma(s+w)},$$

since in each case, by multiplying and dividing the right side of the iterativity condition by the integral which is computed on the left side, and using a suitable substitution, the integral on the right side may be transformed into (7). For example, consider the straight line path in $\mathbb{P}^1\backslash\{0,1,\infty\}$ between two points x and b. On the left side of (3) we arrive at

$$\frac{1}{\Gamma(s+w)}\left[\log\left(\frac{b}{x(1-t)+bt}\right)\right]^{s+w-1},$$

so multiplying and dividing the right side of (3) by the logarithmic factor, we obtain

$$\frac{1}{\Gamma(s)\Gamma(w)}\left[\log\left(\frac{b}{x(1-t)+bt}\right)\right]^{s+w-1}$$

$$\cdot\int_t^1\left(\frac{\log\dfrac{x(1-z)+zb}{x(1-t)+tb}}{\log\dfrac{b}{x(1-t)+tb}}\right)^{s-1}\left(\frac{\log\dfrac{b}{x(1-z)+zb}}{\log\dfrac{b}{x(1-t)+tb}}\right)^{w-1}\frac{1}{\log\dfrac{b}{x(1-t)+tb}}\frac{(b-x)dz}{z(b-x)+x}$$

in which the integral is the same as that in (7) under the substitution

$$a=\frac{\log\dfrac{x(1-z)+zb}{x(1-t)+tb}}{\log\dfrac{b}{x(1-t)+tb}}.$$

For the straight line path $[t,1]$ where $t\in(0,1)$, the iterativity property was known classically, since in this case, the definition accords with the classical fractional integral (see [An, As, Ro]): If x is a real variable with $x\in(0,\infty)$ and $f(t)$ is integrable on this interval, then one notices that by repeated integration by parts, $\frac{1}{(k-1)!}\int_0^x(x-t)^{k-1}f(t)dt$ may be considered to be a k-fold integral of f. For $\operatorname{Re}s>0$, we then define the s-fold integral of f as

$$I_sf(x):=\frac{1}{\Gamma(s)}\int_0^x(x-t)^{s-1}f(t)dt.$$

By the change of variables $-\log z=x-t$, this may be seen to coincide with

$$\int_{[e^{-x},1]}f(x-\log z)\frac{dz}{z}\left(\frac{dz}{z}\right)^{s-1}.$$

Now the additivity property $I_sI_t=I_{s+t}$ of the operator I_s amounts to our iterativity property.

On $\mathbb{P}^1\backslash\{0,1,\infty\}$ we may also take $\beta=\dfrac{dt}{1-t}$, and in integrating over the tangential path $[0,1]$, it is possible to show that such iterated integrals are characterized by the fact that they interpolate those integrals where the iteration occurs an integer number of times, while satisfying a suitable iterative property. We proceed to prove this.

Recall that (4) is valid for any differential 1-form β, and as discussed before it has a unique interpolation once a choice has been made of a branch of the logarithm. Therefore, if an iterative property

can be established in the case of iteration over $\dfrac{dt}{1-t}$ along the path $[0,u]$ for $0<u\le 1$, necessarily

$$\int_{[0,u]}\left(\frac{dt}{1-t}\right)^{s-1}=\frac{1}{\Gamma(s)}\left(\int_{[0,u]}\frac{dt}{1-t}\right)^{s-1}.$$

But then for any k-Bieberbach f,

$$\begin{aligned}\int_{[0,1]} f(z)\left(\frac{dz}{z}\right)^{s-1} &= \int_0^1 \frac{(-\log z)^{s-1}}{\Gamma(s)} f(z)\frac{dz}{z}\\ &= -\int_1^0 \frac{(-\log(1-t))^{s-1}}{\Gamma(s)} f(1-t)\frac{dt}{1-t} \qquad (t=1-z)\\ &= \int_0^1\left(\int_{[0,u]}\left(\frac{dt}{1-t}\right)^{s-1}\right) f(1-u)\frac{du}{1-u}.\end{aligned}$$

We shall use the notation $\int_{[0,1]}\left(\dfrac{dt}{1-t}\right)^{s-1} f(1-t)\dfrac{dt}{1-t}$ for this last integral expression, which is justified by the iterative property to follow.

Recall that the iterativity property is the statement that for a fixed $r\in\mathbb{C}$ with $\mathrm{Re}\,(r)>k+1$, then for any $w\in\mathbb{C}$ with $\mathrm{Re}\,(r)>\mathrm{Re}\,(w)>k+1$, and for any $g(z)$ for which $g(1-z)$ is k-Bieberbach, it follows that

$$\int_{[0,1]}\left(\int_{[0,v]}\left(\frac{dt}{1-t}\right)^{r-w-1}\right)\left(\frac{dt}{1-t}\right)^{w} g(t)\frac{dt}{1-t}=\int_{[0,1]}\left(\frac{dt}{1-t}\right)^{r-1} g(t)\frac{dt}{1-t}. \qquad (8)$$

Framed in a different way,

$$\int_0^1\int_0^u \frac{(-\log(1-t))^{r-w-1}}{\Gamma(r-w)}\frac{(\log(1-t)-\log(1-u))^{w-1}}{\Gamma(w)}\frac{dt}{1-t}g(u)\frac{du}{1-u}=\int_0^1\frac{(-\log(1-u))^{r-1}}{\Gamma(r)}g(u)\frac{du}{1-u}.$$

Again this statement is a consequence of the non-trivial classical fact that the beta integral has an expression in terms of values of the gamma function: Indeed, for

$$\frac{(-\log(1-u))^{r-1}}{\Gamma(r)}=\int_0^u\frac{(-\log(1-t))^{r-w-1}}{\Gamma(r-w)}\frac{(\log(1-t)-\log(1-u))^{w-1}}{\Gamma(w)}\frac{dt}{1-t} \qquad (9)$$

to hold,

$$\beta(w,r-w)=\frac{\Gamma(w)\Gamma(r-w)}{\Gamma(r)}=\int_0^u\left(\frac{\log(1-t)}{\log(1-u)}\right)^{r-w-1}\frac{(\log(1-t)-\log(1-u))^{w-1}}{(-\log(1-u))^{w})}\frac{dt}{1-t}$$

must be true, and it *is* since the substitution

$$\frac{\log(1-t)}{\log(1-u)}=y$$

can be made to show that the integral is the same as

$$\int_0^1 y^{r-w-1}(1-y)^{w-1}dy=\beta(w,r-w).$$

Theorem 1 *For* $(\mathbb{P}^1\backslash\{0,1,\infty\}, \frac{dt}{1-t}, [0,1])$, *Defintion 1 is the only interpolation possible for which the iterativity property (8) holds.*

Proof: The proof of the iterativity property works in this case since the definition implies that

$$G(u,s,w) := \int_0^u \frac{(-\log(1-t))^{s-1}}{\Gamma(s)} \left(\frac{dt}{1-t}\right)^w = \int_0^u \frac{(-\log(1-t))^{s-1}}{\Gamma(s)} \frac{(\log(1-t)-\log(1-u))^{w-1}}{\Gamma(w)} \frac{dt}{1-t}, \tag{10}$$

where we write $s = r - w$. Should some other such integral expression exist, say

$$G(u,s,w) = \int_0^u \frac{(-\log(1-t))^{s-1}}{\Gamma(s)} F_w(t) \frac{dt}{1-t},$$

then for integer $w = n > 1$, in fact

$$F_n(t) = \frac{(\log(1-t)-\log(1-u))^{n-1}}{(n-1)!}$$

because the usual antipode property may be used. For $F_w(t)$ to be a function in w interpolating the $F_n(t)$, necessarily

$$F_w(t) = e^{2\pi ir} \frac{(\log(1-t)-\log(1-u))^{w-1}}{\Gamma(w)}$$

by the considerations pertaining to complex powers discussed before, for some integer r . Of course, here $r = 0$ since there is no such exponential factor in the known expression for $G(u,s,w)$ in (10). $\square$

2.1 Multiplicative iterativity

The above development of the complex iterated integral takes as departure point the iterative property which is necessarily satisfied. When iterating $\frac{dz}{z}$ over the tangential path $[0,1]$ in $\mathbb{P}^1\backslash\{0,1,\infty\}$, however, a second iterativity property holds.

In order to show this, we firstly prove an important fact which follows as a simple consequence of another computationally useful property, namely the power invariance of the iterated integral:

Proposition 1 (Haar Property)
Suppose that $f(z)$ *is* k *-Bieberbach. Let* α *denote a positive real number. Then*

$$\int_{[0,1]} f(z^\alpha) \left(\alpha \frac{dz}{z}\right)^s = \int_{[0,1]} f(z) \left(\frac{dz}{z}\right)^s .$$

As the notation suggests,

$$\int_{[0,1]} g(z) \left(\alpha \frac{dz}{z}\right)^s := \int_0^1 \frac{(-\alpha \log z)^{s-1}}{\Gamma(s)} g(z) \alpha \frac{dz}{z} = \alpha^s \int_0^1 g(z) \left(\frac{dz}{z}\right)^s .$$

Motivating the term 'power invariance' is the fact that

$$d \log z^\alpha = \frac{dz^\alpha}{z^\alpha} = \frac{\alpha z^{\alpha-1} dz}{z^\alpha} = \alpha \frac{dz}{z},$$

and in the coordinates on $\mathbb{C}$ this amounts to invariance under the multiplicative group, hence the reference to Haar measures.

Proof: Again it suffices to show the statement for $f(z) = z^k$ where k is a non-negative integer. In this case, a direct computation involving a substitution $z^{\alpha k} = v$ shows the left side to equal

$$\frac{1}{k^s}.$$

Then observe that this is the value of the right side of the equation we are proving, via the substitution $z^k = u$. $\square$

Then we have the

Corollary 0 *For integer* $k > 0$ *and* $\operatorname{Re} s > 1$ *,*

$$\int_{[0,1]} z^k \left(\frac{dz}{z}\right)^s = \frac{1}{k^s}.$$

Proof: Take $f(z) = z$ and $\alpha = k$ in the HAAR property. Then use the fundamental fact, which is essentially equivalent to the definition of the Gamma function, that for any s,

$$\int_{[0,1]} z \left(\frac{dz}{z}\right)^s = 1.$$

$\square$

Now for notational ease make the

Definition 3 *If* $F(z) = \sum_{n=-m}^{\infty} a_n z^n$ *and* $\operatorname{Re} s > 1$, *define the* s *-gap transform of* F *to be*

$${}^sF(z) := \sum_{n=-m}^{\infty} a_n z^{n^s}.$$

The following result furnishes an alternative means of defining complex iterated integrals of certain functions:

Theorem 2 *If* $F(z) = \sum_{n=0}^{\infty} a_n z^n$ *is at least* k *-BIEBERBACH for some* $k \geq 0$ *and* $\operatorname{Re} s > k+1$, *then*

$$\int_{[0,1]} F(z) \left(\frac{dz}{z}\right)^s = \int_0^1 {}^sF(z) \frac{dz}{z}.$$

Proof: From Corollary 0, when $n \neq 0$,

$$\int_0^1 z^{n^s} \frac{dz}{z} = \frac{1}{n^s},$$

while $\int_0^1 1 \frac{dz}{z} = 0$ by the regularizations we are using. But then invoking the interchange of the sum and integral by means of ideas as in the proof of Lemma 0, it follows that both sides of the equation give $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$. $\square$

Then we have:

Theorem 3 [Multiplicative Iterative Property] *If* $F(z) = \sum_{n=1}^{\infty} a_n z^n$ *is* r-BIEBERBACH *for some* $r > 0$, *then for integer* $k \geq 1$, ${}^k F(z)$ *is* $\dfrac{r}{k}$-BIEBERBACH *and for* $\mathrm{Re}\, s > r + k$,

$$\int_{[0,1]} F(z) \left(\frac{dz}{z}\right)^{s} = \int_{[0,1]} {}^k F(z) \left(\frac{dz}{z}\right)^{\frac{s}{k}}.$$

Proof: That ${}^k F(z)$ is $\dfrac{r}{k}$-BIEBERBACH is a triviality, and the equality of integrals follows from

$$\int_{[0,1]} z^{n^k} \left(\frac{dz}{z}\right)^{\frac{s}{k}} = \frac{1}{(n^k)^{\frac{s}{k}}} = \frac{1}{n^s},$$

where we are again using Corollary 0. □

We remark that the failure of the expression for ${}^s F(z)$ given in the definition of the s-gap transform to be a power series for general non-integer s precludes the proof of a more general iterative property without use of a vastly more complicated approach.

2.2 The RIEMANN zeta function as an iterated integral

Using the formalism of the (additive) iterated integrals, ABEL'S integral

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^s}{e^x - 1} \frac{dx}{x} \tag{11}$$

may be thought of as an s-iterated integral:

Theorem 4 *Whenever* $\mathrm{Re}\,(s) > 1$, *then*

$$\zeta(s) = \int_{[0,1]} \frac{dz}{1-z} \left(\frac{dz}{z}\right)^{s-1}. \tag{12}$$

Proof: In (11), make the substitution $-\log z = x$ and invoke the definition of the iterated integrals.

Alternately, the statement is immediate from a direct computation using Corollary 0 (with k assuming successive positive integer values over which the sum is then taken). □

A related family of complex iterated integrals for the RIEMANN zeta function emerges from use of multiplicative iterativity:

$$\int_{[0,1]} \sum_{n=1}^{\infty} z^{n^k} \left(\frac{dz}{z}\right)^{\frac{s}{k}} = \int_{[0,1]} \sum_{n=1}^{\infty} z^n \left(\frac{dz}{z}\right)^{s} = \int_{[0,1]} \frac{z}{1-z} \left(\frac{dz}{z}\right)^{s} = \zeta(s)$$

for any integer $k \geq 1$. Notice that when $k = 2$, we recover the integral for $\zeta(s)$ involving the theta function

$$\zeta(s) = \frac{1}{\pi^{\frac{-s}{2}} \Gamma(\frac{s}{2})} \int_0^{\infty} x^{\frac{s}{2}} \sum_{n=1}^{\infty} e^{-\pi x n^2} \frac{dx}{x},$$

by the change of variables $z = e^{-\pi x}$. The multiplicative iterativity of this integral expression contrasts with the additive iterativity of RIEMANN'S integral (11). This is very interesting if one remembers that RIEMANN gave two proofs of the functional equation for $\zeta(s)$, one using each of these integral expressions. However, he does not seem to have been aware of any complementarity in these perspectives, or if he was makes no mention thereof.

We remark that this discussion could be carried out with iteration over $\dfrac{dt}{1-t}$ instead. Then for example

$$\zeta(s) = \int_{[0,1]} \left(\frac{dt}{1-t}\right)^{s-1} \frac{dt}{t}.$$

Including the local zeta function $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$, (i.e. the (local) zeta function at the real prime), as a factor with $\zeta(s)$, we obtain the function $Z(s)$ which may also be expressed as an s-iterated integral as follows:

Theorem 5 *For* $\mathrm{Re}\, s > 1$,

$$Z(s) = \int_{[0,1]} \frac{dx}{1-x} \left(\frac{1}{2\pi}\frac{(-\log x)dx}{x}\right)^{\frac{s-1}{2}}$$

Lemma 1 *(Doubling formula of* LEGENDRE *for the factorial function).*

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z+\frac{1}{2}).$$

Proof of Theorem:

$$\begin{aligned}
Z(s) &= \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) \\
&= \frac{\pi^{-s/2}\Gamma(\frac{s}{2})}{\Gamma(s)} \int_0^1 \frac{(-\log x)^{s-1}dx}{1-x} \\
&= \frac{\pi^{(1-s)/2}2^{1-s}}{\Gamma(\frac{s-1}{2}+1)} \int_0^1 (-1)^{(s-1)/2} \frac{(-(\log x)^2)^{(s-1)/2}dx}{1-x} \\
&= \frac{(-1)^{(s-1)/2}}{2^{(s-1)/2}(2\pi)^{(s-1)/2}\Gamma(\frac{s-1}{2}+1)} \int_0^1 \left(\int_1^x -\frac{2\log x dx}{x}\right)^{(s-1)/2} \frac{dx}{1-x} \\
&= \int_{[0,1]} \frac{dx}{1-x} \left(-\frac{\log x dx}{2\pi x}\right)^{(s-1)/2} \qquad \square
\end{aligned}$$

Recall that the functional equation of the RIEMANN zeta function is the statement that this function $Z(s)$ is invariant under the transformation $s \mapsto 1-s$.

We remark that it is possible to develop DIRICHLET L-functions as iterated integrals using similar ideas. In particular, we find that for a character χ of conductor f,

$$L(s,\chi) = \int_{[0,1]} \frac{\sum_{a=1}^{f} \chi(a) z^a}{1-z^f} \left(\frac{dz}{z}\right)^s,$$

which may be developed using the multiplicativity property as

$$L(s,\chi) = \int_{[0,1]} \sum_{a=1}^{f} \chi(a) \sum_{n=0}^{\infty} z^{(a+nf)^k} \left(\frac{dz}{z}\right)^{\frac{s}{k}}$$

for any positive integer k.

2.3 Multiple iterated integrals, polyzeta functions and polylogarithms

The formalism may be extended to multiple versions of the iterated integrals by a simple induction argument based on the above. We perform this generalization in the case of the iteration of $\frac{dz}{z}$. For $j=1,2$ consider k_j-BIEBERBACH functions $f_j(z)$ which are holomorphic at $z=0$ with also $f_1(0)=0$. Let $s_j \in \mathbb{C}$ have $\operatorname{Re} s_j > (k_j + j)$. Then

$$\int_{[0,1]} f_1(z)\frac{dz}{z}\left(\frac{dz}{z}\right)^{s_1-1} f_2(z)\frac{dz}{z}\left(\frac{dz}{z}\right)^{s_2-1},$$

which is interpreted as

$$\int_{[0,1]} \left[\int_{[0,u]} f_1(z)\frac{dz}{z}\left(\frac{dx}{x}\right)^{s_1-1}\right] f_2(u)\frac{du}{u}\left(\frac{du}{u}\right)^{s_2-1}$$

converges by a similar argument to the one given before.[The vanishing of $f_1(z)$ at zero (so that the LAURENT series has first non-zero coefficient that of the linear term) facilitates the proof since we can use the bound

$$\left|a_1 b_{m-1} + \frac{a_2 b_{m-2}}{2^{s_1}} + \ldots + \frac{a_m b_0}{m^{s_1}}\right| \leq m C_1 C_2 m^{k_2}$$

where the a_j are coefficients for the power series for f_1 and the b_j for the LAURENT series for f_2; the C_1 factor bounds the coefficients $|\frac{a_j}{j^{s_1}}|$ with $0 < j \leq m$ (taking

$$C_1 = \max\{|\frac{a_j}{j^{s_1}}| : 1 \leq j < N_{k_1}\} \cup \{C_{k_1}\}$$

with notation as in the definition of k-BIEBERBACH functions), and the $C_2 m^{k_2}$ bound the $|b_j|$. As seen before, the integral then converges provided $\operatorname{Re} s_2 > (k_2+2)$.] Here,

$$\int_{[0,u]} f_1(z)\frac{dz}{z}\left(\frac{dz}{z}\right)^{s_1-1} = \int_0^u \frac{(\log u - \log z)^{s_1-1}}{\Gamma s_1} f_1(z)\frac{dz}{z} =: h_1(u,s_1)$$

is a complex iterated integral which satisfies an iterative property: With notation as above (and polynomial $p(x)$),

$$\begin{aligned}
\int_{[0,u]} p(x)\left(\frac{dx}{x}\right)^r &= \int_0^u \frac{(\log u - \log x)^{r-1}}{\Gamma(r)} p(x)\frac{dx}{x} \\
&= \int_0^u \int_0^{u_1} \frac{(\log u - \log u_1)^{r-w-1}}{\Gamma(r-w)} \frac{(\log u_1 - \log x)^{w-1}}{\Gamma(w)} p(x)\frac{dx}{x}\frac{du_1}{u_1} \\
&= \int_{[0,u]} p(x)\left(\frac{dx}{x}\right)^w \left(\frac{du_1}{u_1}\right)^{r-w}
\end{aligned}$$

follows from linearity by means of the substitution $v = \dfrac{x}{u}$ in the second expression, use of the iterativity property (2), and then the substitution $u_1 = u\tilde{u}$ for some intermediate variable $\tilde{u}$, followed by the reverse substitution $x = uv$.

But then

$$\int_{[0,1]} f_1(z)\frac{dz}{z}\left(\frac{dz}{z}\right)^{s_1-1} f_2(z)\frac{dz}{z}\left(\frac{dz}{z}\right)^{s_2-1} = \int_{[0,1]} h(z,s_1)f_2(z)\frac{dz}{z}\left(\frac{dz}{z}\right)^{s_2-1},$$

where $h(z,s_1)f_2(z)$ is (k_2+1)-Bieberbach so that the iterative property holds not only in s_1, but in s_2 as well.

These ideas motivate the

Definition 4 *Suppose that* $\mathbf{f} := (f_1(z),\ldots,f_l(z))$ *is a tuple of functions each holomorphic at* $z=0$ *with* $f_1(0)=0$*, such that* $f_j(z)$ *is* k_j*-*Bieberbach*. Then the* $\mathbf{s} := (s_1,\ldots,s_l)$*-multiple iterated integral of* $\mathbf{f}$ *against* $\dfrac{dz}{z}$ *is*

$$\begin{aligned}
&\int_{[0,1]} f_1(z)\frac{dz}{z}\left(\frac{dz}{z}\right)^{s_1-1}\cdots f_l(z)\frac{dz}{z}\left(\frac{dz}{z}\right)^{s_l-1}\\
:=\ &\int_0^1\int_0^{t_l}\cdots\int_0^{t_2}\frac{(\log t_2-\log t_1)^{s_1-1}}{\Gamma(s_1)}f_1(t_1)\frac{dt_1}{t_1}\cdots\\
&\frac{(\log t_l-\log t_{l-1})^{s_{l-1}-1}}{\Gamma(s_{l-1})}f_{l-1}(t_{l-1})\frac{dt_{l-1}}{t_{l-1}}\frac{(-\log t_l)^{s_l-1}}{\Gamma(s_l)}f_l(t_l)\frac{dt_l}{t_l}
\end{aligned}$$

provided that $\mathrm{Re}\,(s_j) > (k_j+j)$ *for* $1\le j\le l$.

Continuing the argument preceding the definition inductively, we establish the

Theorem 6 (Multiple Iterative Property) *For a fixed* $\mathbf{r} = (r_1,\ldots,r_l)\in\mathbb{C}^l$ *with* $\mathrm{Re}\,(r_j) > k_j+j$ *for* $1\le j\le l$ *then for any* $(s_1,\ldots,s_l)\in\mathbb{C}^l$ *with* $\mathrm{Re}\,(r_j) > \mathrm{Re}\,(s_j) > k_j+j$ *for* $1\le j\le l$*, and for any tuple* $(f_1(z),\ldots,f_l(z))$ *of functions each holomorphic at* $z=0$*, with* $f_1(z)$ *vanishing at* $z=0$*, and with* $f_j(z)$ *being* k_j*-*Bieberbach*, it follows that writing* $w_j = r_j - s_j$*, we have*

$$\int_{[0,1]} f_1(z)\left(\frac{dz}{z}\right)^{s_1}\left(\frac{dz}{z}\right)^{w_1}\cdots f_l(z)\left(\frac{dz}{z}\right)^{s_l}\left(\frac{dz}{z}\right)^{w_l} = \int_{[0,1]} f_1(z)\left(\frac{dz}{z}\right)^{r_1}\cdots f_l(z)\left(\frac{dz}{z}\right)^{r_l}.$$

Otherwise stated,

$$\begin{aligned}
&\int_0^1\int_0^{\tilde{t}_l}\int_0^{t_l}\cdots\int_0^{t_2}\int_0^{\tilde{t}_1}\frac{(\log\tilde{t}_1-\log t_1)^{s_1-1}}{\Gamma(s_1)}f_1(t_1)\frac{dt_1}{t_1}\frac{(\log t_2-\log\tilde{t}_1)^{w_1-1}}{\Gamma(w_1)}\frac{d\tilde{t}_1}{\tilde{t}_1}\\
&\cdots\frac{(\log t_l-\log\tilde{t}_{l-1})^{w_{l-1}-1}}{\Gamma(w_{l-1})}\frac{d\tilde{t}_{l-1}}{\tilde{t}_{l-1}}\frac{(\log\tilde{t}_l-\log t_l)^{s_l-1}}{\Gamma(s_l)}f_l(t_l)\frac{dt_l}{t_l}\frac{(-\log\tilde{t}_l)^{w_l-1}}{\Gamma(w_l)}\frac{d\tilde{t}_l}{\tilde{t}_l}\\
&=\int_0^1\int_0^{u_l}\cdots\int_0^{u_2}\frac{(\log u_2-\log u_1)^{r_1-1}}{\Gamma(r_1)}f_1(u_1)\frac{du_1}{u_1}\\
&\cdots\frac{(\log u_l-\log u_{l-1})^{r_{l-1}-1}}{\Gamma(r_{l-1})}f_{l-1}(u_{l-1})\frac{du_{l-1}}{u_{l-1}}\frac{(-\log u_l)^{r_l-1}}{\Gamma(r_l)}f_l(u_l)\frac{du_l}{u_l}.
\end{aligned}$$

Next we explain how the polyzeta functions may be expressed as complex iterated integrals. For integers n_j, it is well-known that the polyzeta numbers (also referred to as multiple zeta values in the literature), may be expressed as $(n_1+\ldots+n_k)$-fold iterated integrals

$$\zeta(n_1,\ldots,n_k)=\int_{[0,1]}\frac{dz}{1-z}\left(\frac{dz}{z}\right)^{n_1}\frac{dz}{1-z}\left(\frac{dz}{z}\right)^{n_2}\cdots\frac{dz}{1-z}\left(\frac{dz}{z}\right)^{n_k}.$$

Once again, this expression also makes sense when the n_j are replaced by non-integral complex numbers s_j.

Goncharov and Kontsevich found the following integral representation for the polyzeta functions:

$$\zeta(s_1,s_2,\ldots,s_l)=\frac{1}{\Gamma(s_1)}\frac{1}{\Gamma(s_2)}\cdots\frac{1}{\Gamma(s_l)}\int_0^\infty\cdots\int_0^\infty\int_0^\infty\frac{t_1^{s_1-1}dt_1}{e^{t_1+t_2+\ldots+t_l}-1}\frac{t_2^{s_2-1}dt_2}{e^{t_2+t_3+\ldots+t_l}-1}\cdots\frac{t_l^{s_l-1}dt_l}{e^{t_l-1}-1},\tag{13}$$

valid provided $\mathrm{Re}\,(s_{l-j+1}+\ldots+s_l)>j$ for $1\le j\le l$. When $l=1$, the integral is the same as the expression for Riemann's zeta function found by Abel. ([Hi] is an excellent reference for this and other classical formulas we require.)

It happens that using the $\mathbb{P}^1\backslash\{0,1,\infty\}$ coordinates $x_1,\ldots,x_k$ determined via $x_1=e^{t_1+\ldots+t_k}$ and $x_{j+1}=e^{-t_j}x_j$ for $j=1,\ldots,k-1$, the integral is

$$\int_0^1\int_0^{x_k}\cdots\int_0^{x_2}\frac{(\log x_2-\log x_1)^{s_1-1}}{\Gamma(s_1)}\frac{dx_1}{1-x_1}\cdots\frac{(\log x_k-\log x_{k-1})^{s_{k-1}-1}}{\Gamma(s_{k-1})}\frac{dx_{k-1}}{1-x_{k-1}}\frac{(-\log x_k)^{s_k-1}}{\Gamma(s_k)}\frac{dx_k}{1-x_k}\tag{14}$$

which may be regarded in an obvious way as a k-fold iterated integral along $[0,1]$ (in the sense of Chen) generalizing Abel's integral. But comparing this to Definition 4 it is clear that in fact

$$\zeta(s_1,\ldots,s_l)=\int_{[0,1]}\frac{dz}{1-z}\left(\frac{dz}{z}\right)^{s_1-1}\frac{dz}{1-z}\cdots\left(\frac{dz}{z}\right)^{s_l-1}$$

whenever $Re(s_j)>j$ for all $1\le j\le l$.

A striking duality exists: As before iteration over $\dfrac{dz}{1-z}$ could also be developed. This would give

$$\zeta(s_1,\ldots,s_l)=\int_{[0,1]}\left(\frac{dz}{1-z}\right)^{s_l-1}\frac{dz}{z}\cdots\left(\frac{dz}{1-z}\right)^{s_1-1}\frac{dz}{z}.$$

We remark that use of a similar change of coordinates for the integral expression known for the polylogarithm functions (see [Ca]) and use of these same ideas yields

$$Li_{(s_1,\ldots,s_l)}(t)=\int_{[0,t]}\frac{dz}{1-z}\left(\frac{dz}{z}\right)^{s_1-1}\frac{dz}{1-z}\cdots\left(\frac{dz}{z}\right)^{s_l-1}$$

for any $t\in[0,1]$, which also holds provided $Re(s_j)>j$ for all $1\le j\le l$. The same formula is valid when $[0,t]$ is taken to indicate the straight line path from 0 to any t in the open unit disc.

Also, multiple versions of the Hurwitz zeta functions may be defined, and by similar considerations these satisfy

$$\begin{aligned}\zeta(s_1,\ldots,s_l;z)\quad&:=\quad\sum_{0<n_1<\ldots<n_l}\frac{1}{(z+n_1)^{s_1}\ldots(z+n_l)^{s_l}}\\&=\quad\int_{[0,1]}\frac{x^{z-1}dx}{1-x}\left(\frac{dx}{x}\right)^{s_1-1}\frac{dx}{1-x}\left(\frac{dx}{x}\right)^{s_2-1}\cdots\frac{dx}{1-x}\left(\frac{dx}{x}\right)^{s_l-1}\end{aligned}$$

whenever $Re(s_j) > j$ for all $1 \leq j \leq l$. Notice that $\zeta(s_1, \ldots, s_l; 1) = \zeta(s_1, \ldots, s_l)$.

The integral expressions for the polyzeta and HURWITZ zeta functions may be thought of as homotopy functionals evaluated along the homotopy class of the path $[0,1]$ in the fundamental groupoid consisting of homotopy classes of paths in $\mathbb{P}^1 \backslash \{0, 1, \infty\}$ from the tangential basepoint $\overrightarrow{01}$ to the tangential basepoint $\overrightarrow{10}$. This particular path is very important, since it is identified with the DRINFEL'D associator Φ under the isomorphism of the unipotent completion of this fundamental groupoid with complex coefficients, with the group-like elements under comultiplication, of the completion of the free associative algebra generated by two symbols over $\mathbb{C}$ (i.e. the algebra of non-commuting power series in two variables, say A_0 and A_1, with complex coefficients).

3 Further applications

3.1 Irrationality of DEDEKIND zeta functions

Definition 5 *When $F(z)$ is some k-BIEBERBACH function, we shall call*

$$L(F)(s) := \int_{[0,1]} F(z) \left(\frac{dz}{z}\right)^s$$

the L-function of F.

Then the L-function of $F_{\mathbb{Q}}(z) = \dfrac{z}{1-z}$ is the RIEMANN zeta function, and we notice immediately that $L(F_{\mathbb{Q}})(s) = \zeta(s)$ has a simple pole at $s = 1$ while $F_{\mathbb{Q}}(z)$ has a simple pole at $z = 1$.

On the other hand, if χ is a non-trivial DIRICHLET character of conductor f, the L-function of

$$F_\chi(z) := \sum_{a=1}^{f} \frac{\chi(a) z^a}{1 - z^f}$$

is the DIRICHLET L-function $L(s, \chi) = \zeta(s, 0; \chi)$ of χ, but in this case, $F_\chi(z)$ has no pole at $z = 1$ and $L(F_\chi)(s)$ is non-singular at $s = 1$.

Now z is a coordinate on $\mathbb{P}^1 \backslash \{0, 1, \infty\}$ while s describes $\mathbb{C}$, so there is *a priori* no connexion between them. For this reason, the correspondence between a pole of a function of z and a pole of an associated function of s in the case of DIRICHLET L-functions may appear somewhat surprising. It turns out to be a consequence of the existence of the analytic continuation for the L-functions in the style of RIEMANN'S integral expression giving the analytic continuation of $\zeta(s)$. As such, this correspondence holds quite generally:

Theorem 7 *Suppose that $F(z)$ is k-BIEBERBACH for some k, and is meromorphic in some neighborhood of $z = 1$. Then $L(F)(s)$ has a pole at $s = 1$ if and only if $\dfrac{F(z)}{z}$ has a pole of non-zero residue at $z = 1$. Moreover, any pole of $L(F)(s)$ at $s = 1$ is simple. When the pole of $F(z)$ at $z = 1$ is also simple, the residue agrees with that of $L(F)(s)$ at $s = 1$. More generally, if $F(z) = \sum_{n \geq -m} a_n (z-1)^n$, the residue of $L(F)(s)$ at $s = 1$ is $\sum_{n=-m}^{-1} (-1)^{1-n} a_n$.*

Proof: There exists $m \geq 1$ for which $G_m(x) := x^m F(e^{-x})$ is regular at $x = 0$. We now fix m as follows: If $F(z)$ is regular at $z = 1$, take $m = 1$. Otherwise, let $m \geq 1$ be minimal such that $G_m(0) \neq 0$ but $G_m(x)$ is regular at $x = 0$.

Then define

$$H(F)(s) = \int_C (-x)^{s-1} F(e^{-x}) dx$$

where C once again denotes the RIEMANN contour, and consider $H(F)$ for $\mathrm{Re}\,(s) > m+k+1$. Denoting the part of C which is a loop about zero by γ_0,

$$\int_{\gamma_0} x^m F(e^{-x}) \frac{x^s}{x^m} \frac{dx}{x} = 0$$

by CAUCHY'S integral theorem. Consequently, as in the proof of Theorem 9, we find that

$$H(F)(s) = (e^{i\pi s} - e^{-i\pi s})\Gamma(s) L(F)(s)$$

for all s with $\mathrm{Re}\,(s) > m+k+1$ and hence on all of $\mathbb{C}$.

Then again as in the proof of Theorem 9, we have

$$L(F)(s) = \frac{1}{\Gamma(s) 2i \sin(\pi s)} H(F)(s) = \frac{\Gamma(1-s)}{2\pi i} H(F)(s).$$

But

$$H(F)(1) = \int_{\gamma_0} F(e^{-x}) dx$$

because the integrals along the real axis cancel each other out. Now by the residue theorem the integral is non-zero exactly when $F(e^{-x})$ has a pole of non-zero residue at $x=0$, which is precisely when $\frac{F(z)}{z}$ has a pole of non-zero residue at $z=1$. Such are the instances in which $L(F)(s)$ has a simple pole at $s=1$.

Since the residue of $\Gamma(1-s)$ at $s=1$ is 1, we also see that the residue of $L(F)$ at $s=1$ is

$$\frac{1}{2\pi i} \cdot 2\pi i \mathrm{Res}_{x=0} F(e^{-x}) = \frac{1}{2\pi i} \int_{\gamma_1} F(z) \frac{dz}{z}$$

where γ_1 is a positively oriented loop about $z=1$. Using the power series expansion for $\frac{1}{z}$ at $z=1$, the statement about the residues follows. $\square$

Suppose now that K is a number field of degree N over $\mathbb{Q}$ and I denotes the set of non-zero integral ideals of K. Consider the DEDEKIND zeta function $\zeta_K(s)$, which is known to have a simple pole at $s=1$. This function also has a complex iterated integral expression, as may be seen from the

Lemma 2 *The power series*

$$F_K(z) := \sum_{\mathfrak{a} \in I} z^{N(\mathfrak{a})}$$

is at least 1-BIEBERBACH.

Proof: Notice that if $\nu(n)$ denotes the number of ideals of I of norm equal to n, we have

$$\sum_{\mathfrak{a} \in I} z^{N(\mathfrak{a})} = \sum_{n=1}^{\infty} \nu(n) z^n.$$

Now $\sum_{j=1}^m \nu(j) = \rho_K m + O(m^{1-\frac{1}{N}})$ where ρ_K is the residue of $\zeta_K(s)$ at $s=1$. (See [La], for example.) The rough estimate $\nu(n) \le Cn$ then suffices to prove the lemma. 2cm $\square$

Convergence of the power series is uniform on compacta in the disc. Hence we may write

$$\int_{[0,1]} \sum_{\mathfrak{a}\in I} z^{N(\mathfrak{a})} \left(\frac{dz}{z}\right)^s = \sum_{\mathfrak{a}\in I} \frac{1}{N(\mathfrak{a})^s} =: \zeta_K(s)$$

for $\mathrm{Re}\,(s) > 2$.

This suggests an archimedean analogue of the Iwasawa algebra: The zeta function of a number field can be viewed as a power series in

$$\Lambda_\infty := \mathbb{Z}[[T]].$$

How far this analogy with Iwasawa theory can be taken is an interesting question. One would like to see that properties of the L-function of a power series are reflected in those of the power series itself.

For example, the function

$$F_{pr}(z) := \sum_{p \text{ prime}} z^p$$

is not analytically continuable beyond the boundary of the unit disk, by the Fabry gap theorem. This ought to reflect properties of

$$L(F_{pr})(s) = \sum_{p \text{ prime}} \frac{1}{p^s}.$$

As another example, consider the power series $\sum_{n=1}^{\infty} \mu(n) z^n$, which is known to be singular at $z = 1$. (See [Fa].) The iterated integral of this function is $\dfrac{1}{\zeta(s)}$, so by Theorem 7 we see that $\sum_{n=1}^{\infty} \mu(n) z^n$ is not holomorphic on any punctured neighborhood of $z = 1$.

Using the ideas of the previous section, it is evident that the rationality of the values of the Riemann zeta function at negative integers is a direct consequence of the fact that $F_{\mathbb{Q}}(z) = \dfrac{z}{1-z}$ is rational. In line with the general philosophy that "zeta functions should be rational" we might expect that $F_K(z)$ would also be rational, but this is not true in certain cases:

From Theorem 7, because $\zeta_K(s)$ has a pole at $s = 1$, we know that $F_K(z)$ is not regular at $z = 1$. Should $F_K(z)$ have a pole there, in the Laurent series expansion for $F_K(z)$ at $z = 1$, a finite (alternating) sum of coefficients of $F_K(z)$ would be equal to the residue of $\zeta_K(s)$ at $s = 1$, which is known to be given by

$$\rho_K = \frac{2^{r_1}(2\pi)^{r_2} R_K}{w\sqrt{|d_K|}} h_K$$

where r_1 denotes the number of real embeddings of K; $2r_2$ the number of complex embeddings; R_K is the regulator; h_K the class number; d_K the discriminant; and w the number of roots of unity in K.

Now by a theorem of Petersson (see [Bi]), any power series with integer coefficients about zero having radius of convergence 1 is either not analytically continuable beyond the boundary of the unit disk or gives rise to a rational function on $\mathbb{C}$; while a theorem of Fatou asserts that such a power series yields a function which is either rational or non-algebraic. Should $F_K(z)$ be a rational function, it would have an expression as a ratio of polynomials with integer coefficients by an elementary argument given in [Bi]: $F_K(z)$ is expressible as a power series with integer coefficents on the unit disk; so with notation $\nu(n)$ as above, if

$$\sum_{n=0}^{m} p_n z^n \cdot \sum_{n=1}^{\infty} \nu(n) z^n = \sum_{n=0}^{l} q_n z^n,$$

then infinitely many linear equations with integer coefficients $\nu(n)$ arise, among which there is a solution in integers given that some solution exists. In this way, a LAURENT series expansion about $z = 1$ with rational coefficients would ensue. By Theorem 7, some linear combination of certain of these coefficients would have to equal ρ_K. (The only singularities of rational functions are poles.) But ρ_K is expected to always be irrational, and whenever it is, $F_K(z)$ could not be rational. (The difficulty in proving irrationality of ρ_K lies in the fact that in general, both π and logarithms of units appear in the formula for ρ_K and it is not obvious that the product of these factors remains transcendental.)

Hence we have

Theorem 8 *For a number field K for which ρ_K is irrational, $F(K)(z)$ is non-algebraic and non-continuable outside of the unit disk.*

Consequently, we can also state the

Corollary 1 *Irrationality of ρ_K is an obstruction to the existence of a contour integral proof of the analytic continuation and functional equation for $\zeta_K(s)$ along the lines of* RIEMANN'S *first proof of the functional equation of $\zeta(s)$.*
This is evident from the fact that the contour of principal interest in such a proof would loop about $z = 1$.

3.2 Iterated integrals and derivatives

EULER conceived of an ingenious way to assign meaning to the divergent infinite sum

$$\sum_{n=1}^{\infty} n^k$$

for $k \geq 1$. The argument uses ABEL summation but ignores the divergence of the series being manipulated.[3]

Now let $a \in \mathbb{N}$ have $a \geq 2$ and define

$$\xi_a(n) = \begin{cases} 1 & \text{if } n \not\equiv 0 \ (a) \\ 1 - a & \text{if } n \equiv 0 \ (a). \end{cases}$$

Also let

$$\Psi(t) = \frac{\sum_{n=1}^{a} \xi_a(n) t^n}{1 - t^a}.$$

Using EULER'S ideas, KATZ produced the following generalization of his formula:

$$\left(t \frac{d}{dt} \right)^m \Psi(t)|_{t=1} = (1 - a^{m+1})\zeta(-m), \tag{15}$$

for positive integers m.

Using the formalism of complex iterated integrals, it is not hard to see that also

$$\int_{[0,1]} \Psi(t) \left(\frac{dt}{t} \right)^s = (1 - a^{1-s})\zeta(s).$$

[3]Perhaps the most surprising fact in connection with this argument is that it gives the same (correct) values of the RIEMANN zeta function at negative integers, as a more rigorous approach does!

whenever $\operatorname{Re} s > 1$.

This remarkable interplay between iterated derivatives and integrals holds quite generally:

Theorem 9 [GEL'FAND - SHILOV] *If $F(t) = \sum_{n=0}^{\infty} a_n t^n$ is holomorphic on the unit disk centered at $t = 0$, and is also analytic in some neighborhood of $t = 1$, then as a function of s,*

$$\int_{[0,1]} F(t) \left(\frac{dt}{t}\right)^s$$

admits an analytic continuation which at negative integers $-k$ is given by

$$\left(t\frac{d}{dt}\right)^k F(t)|_{t=1}$$

Proof: Let $G(x) := F(e^{-x})$ and observe that this function is analytic in a neighborhood of $x = 0$. Consider

$$H(s) := \int_C (-x)^s G(x) \frac{dx}{x}$$

where C is the RIEMANN contour from $+\infty$ to 0 and back avoiding the positive real axis and looping around 0 once in the positive direction. Also define

$$\begin{aligned} L(F)(s) &:= \frac{1}{\Gamma(s)} \int_0^{\infty} x^s F(e^{-x}) \frac{dx}{x} \\ &= \int_{[0,1]} F(t) \left(\frac{dt}{t}\right)^s, \end{aligned}$$

which converges for $\operatorname{Re}(s) > k+1$ if F is k-BIEBERBACH.

Then we can show that $H(s) = 2i \sin(\pi s) \Gamma(s) L(F)(s)$: Indeed, suppose that $\operatorname{Re} s > k+1$. Then on the first piece of the contour C, (above the real axis) we know that $(-x)^s = e^{s \log x - i\pi s}$ whereas along the last piece of the contour (below the real axis) $(-x)^s = e^{s \log x + i\pi s}$. Also, because $\operatorname{Re} s > 1$, the integrand is non-singular at zero, so as the radius of the loop about zero tends to zero, the value of the integral about this circular piece of C also approaches zero. Then

$$\begin{aligned} H(s) &= (-e^{-i\pi s} + e^{i\pi s}) \int_0^{\infty} x^s G(x) \frac{dx}{x} \\ &= 2i \sin(\pi s) \Gamma(s) L(F)(s). \end{aligned}$$

The integral $H(s)$ converges for all complex s, because $F(e^{-x})$ is a power series in e^{-x} having no constant term, so that $F(e^{-x})$ dominates x^s as x approaches infinity. Also, the convergence is uniform on compacta so the function of s determined by H is complex analytic. Hence, using well-known identities satisfied by the Γ function to write

$$L(F)(s) = \frac{\Gamma(1-s)}{2\pi i} \int_C (-x)^s G(x) \frac{dx}{x}, \tag{16}$$

we see that $L(F)(s)$ is a function defined and analytic at all points other than (possibly) the poles of $\Gamma(1-s)$ - i.e. for $s \notin \mathbb{N} \backslash \{0\}$.

From the convergence of $L(F)(s)$ on some right half-plane in $\mathbb{C}$, we know then that the function has at most finitely many poles - to wit, at integers $0, 1, \leq k+1$.

Consequently it certainly makes sense to investigate the value of $L(F)(s)$ at negative integers, which we proceed to do:

$G(x) = F(e^{-x})$ is analytic in some neighborhood of $0 \in \mathbb{C}$. Then write $G(x) = \sum_{m=0}^{\infty} b_m \dfrac{x^m}{m!}$. On the pieces of the Riemann contour lying above and below the real axis, we again have that $(-x)^{-k} = e^{-k\log x}e^{-i\pi k}$ and $(-x)^k = e^{-k\log x}e^{+i\pi k}$ respectively. Thus the integrals along these pieces are identical, although opposite in sign since the paths run in opposite directions. Hence

$$
\begin{aligned}
L(F)(-k) &= \frac{\Gamma(1+k)}{2\pi i}\int_C (-x)^{-k}G(x)\frac{dx}{x} \\
&= \frac{\Gamma(1+k)}{2\pi i}\left(\int_{+\infty}^{0}(-x)^{-k}G(x)\frac{dx}{x} + \int_{|x|=\delta}(-x)^{-k}G(x)\frac{dx}{x} + \int_{0}^{+\infty}(-x)^{-k}G(x)\frac{dx}{x}\right) \\
&= \frac{k!}{2\pi i}\int_{|x|=\delta}(-x)^{-k}\sum_{m=0}^{\infty} b_m\frac{x^m}{m!}\frac{dx}{x} \\
&= \frac{(-1)^k k!}{2\pi i}\sum_{m=0}^{\infty}\frac{b_m}{m!}\int_{|x|=\delta} x^{m-k}\frac{dx}{x} \text{ from uniform convergence of the sum} \\
&= (-1)^k k!\sum_{m=0}^{\infty}\frac{b_m}{m!}\frac{1}{2\pi}\int_0^{2\pi} x^{m-k}d\theta \\
&= (-1)^k k!\sum_{m=0}^{\infty}\frac{b_m}{m!}\frac{1}{2\pi}(2\pi\delta_{m,k}) \\
&= (-1)^k b_k
\end{aligned}
$$

At the same time,

$$
\begin{aligned}
\left(t\frac{d}{dt}\right)^k F(t)|_{t=1} &= \left(-\frac{d}{dx}\right)^k F(e^{-x})|_{x=0} \\
&= (-1)^k\left(\frac{d}{dx}\right)^k \sum_{m=0}^{\infty} b_m\frac{x^m}{m!}|_{x=0} \\
&= (-1)^k b_k = L(F)(-k)
\end{aligned}
$$

□

The above theorem was expressed by Gel'fand and Shilov in terms of generalized functions - in particular, they show that the normalized distribution $\dfrac{x_+^{s-1}}{\Gamma(s)}$ satisfies

$$\frac{x_+^{s-1}}{\Gamma(s)}|_{s=-n} = \delta^n(x)$$

where

$$\int_0^{\infty}\delta^n(x)\phi(x)dx = \phi^n(0)$$

for any test function ϕ. (See [GS]I.§3.5). This is the same statement as that given above, under the co-ordinate change $x = -\log t$.

The proof of Theorem 9 may easily be modified to show

Theorem 10 *For* F *as above and* $w \in (0,1)$ *arbitrary, then the function*

$$\int_{[w,1]} F(t) \left(\frac{dt}{t}\right)^{s}$$

has the same *analytic continuation to negative integers as does*

$$\int_{[0,1]} F(t) \left(\frac{dt}{t}\right)^{s}.$$

This w-independence is quite surprising. From the distribution viewpoint, it is certainly true that the analytic continuation at negative integers is some kind of derivative DIRAC distribution centered at zero (corresponding to $1 \in \mathbb{P}^1 \backslash \{0,1,\infty\}$), but for $Re(s) > 1$ the distribution is not even compactly supported! In the homotopy theory case, the natural notion of tangential base-point is in evidence here: The analytic continuation of the iterated integrals is the same for all paths which lie along the tangential path between 0 and 1 in $\mathbb{P}^1 \backslash \{0,1,\infty\}$, which end in the tangential base point $\overrightarrow{01}$; but there is no apparent reason why this should be so and as a function of w the iterated integral is certainly non-constant. Observe that this implies that the p-adic L-functions interpolate values of a *family* of functions at negative integers.

The coproduct formula may also be used to give a proof of Theorem 10.

Theorem 9 may be used to immediately write down the formula for the values of L-functions at negative integers. In particular, if χ is a non-trivial DIRICHLET character of conductor f, then

$$L(-m,\chi) = \left(t\frac{d}{dt}\right)^{m} \frac{\sum_{a=1}^{f} \chi(a) t^{a}}{1-t^{f}}\Bigg|_{t=1}.$$

Moreover, for such a character χ, since $\sum_{a=1}^{f} \chi(a) = 0$ we may use Theorem 9 to see that the generalized HURWITZ zeta function

$$\zeta(s,z;\chi) = \int_{[0,1]} t^{z} \sum_{a=1}^{f} \frac{\chi(a) t^{a}}{1-t^{f}} \left(\frac{dt}{t}\right)^{s}$$

has analytic continuation to the negative integers given by

$$\left(t\frac{d}{dt}\right)^{n} \left(t^{z} \sum_{a=1}^{f} \frac{\chi(a) t^{a}}{1-t^{f}}\right)|_{t=1} = -\frac{B_{n+1,\chi}(z)}{n+1}.$$

Effectively this is a rewriting of the definition of the generalized BERNOULLI polynomials using the generating series under the change of coordinates $t = e^{-w}$.

3.3 Monodromy of polylogarithms

Denoting the straight line path from 0 to $w \in D'(0,1)$ by $[0 \to w]$, then as asserted before,

$$Li_{s}(w) = \int_{[0\to w]} \frac{dx}{1-x} \left(\frac{dx}{x}\right)^{s-1}.$$

This is because

$$\begin{aligned}\int_{[0\to z]} x^k dx \left(\frac{dx}{x}\right)^{s-1} &= \int_{[0,1]} (zt)^k z dt \left(\frac{zdt}{zt}\right)^{s-1} \\ &= z^{k+1}\int_{[0,1]} t^k dt \left(\frac{dt}{t}\right)^{s-1} \\ &= \frac{z^{k+1}}{(k+1)^s}\end{aligned}$$

Or seen another way:

$$\begin{aligned}\int_{[0\to z]} x^k dx \left(\frac{dx}{x}\right)^{s-1} &= \frac{(-1)^{s-1}}{\Gamma(s)}\int_0^z \left(\int_0^w \frac{dx}{x}\right)^{s-1} x^k dx \\ &= \frac{(-1)^{s-1}}{\Gamma(s)}\int_0^z \left(\int_0^r \frac{zdt}{zt}\right)^{s-1} x^k dx \text{ where } w = rz \text{ with } r\in[0,1] \\ &= \frac{(-1)^{s-1}}{\Gamma(s)}\int_0^1 \left(\int_0^r \frac{dt}{t}\right)^{s-1} z^{k+1}t^k dt \ \text{ using } \ x = zt \\ &= z^{k+1}\int_{[0,1]} t^k dt \left(\frac{dt}{t}\right)^{s-1} \\ &= \frac{z^{k+1}}{(k+1)^s}.\end{aligned}$$

The sum of such terms for $k = 1, 2, \ldots$ and the interchange of the sum and integral (which as before is allowed because of an argument involving use of the Lebesgue dominated convergence theorem) results in the above formula for the polylogarithm functions.

A well-known fact with an elegant expression in terms of iterated integrals is the general monodromy theorem:

Theorem 11 *$Li_s(w)$ continued analytically along a loop γ about 1 (i.e. the monodromy of the general polylogarithm function) is*

$$Li_s(w) - \frac{2\pi i}{\Gamma(s)}\log^{s-1}(w) = \int_{[0\to w]} \frac{dx}{1-x}\left(\frac{dx}{x}\right)^{s-1} + \int_\gamma \frac{dx}{1-x}\cdot\int_{[1\to w]}\left(\frac{dx}{x}\right)^{s-1}. \tag{17}$$

Classically, Jonquière's formula was used to effect the proof, but a direct topological proof using the homotopy functional (iterated integral) perspective is desirable since (17) is reminiscent of a coproduct formula in which many terms vanish. We proceed to give such a proof:

Proof: Let $\varepsilon > 0$ be fixed, and denote the circular path of radius ε about 1 by $\gamma_{1,\varepsilon}$. Now notice that $[0\to w]$ is homotopic to the composition of the straight line paths in $\mathbb{P}^1\backslash\{0,1,\infty\}$ from 0 to $1-\varepsilon$ and from $1-\varepsilon$ to w, which will be denoted $[0\to 1-\varepsilon]$ and $[1-\varepsilon\to w]$ respectively. Consequently, the homotopy functional

$$\int_. \frac{dx}{1-x}\left(\frac{dx}{x}\right)^{s-1}$$

evaluated along $[0\to w]$ gives the same value as when it is evaluated along $[0\to 1-\varepsilon]\cup[1-\varepsilon\to w]$. i.e.

$$Li_s(w) = \int_{[0\to w]} \frac{dx}{1-x}\left(\frac{dx}{x}\right)^{s-1} = \int_{[0\to 1-\varepsilon]\cup[1-\varepsilon\to w]} \frac{dx}{1-x}\left(\frac{dx}{x}\right)^{s-1}. \tag{18}$$

Then the analytic continuation of $Li_s(w)$ around 1 may be expressed by

$$\int_{[0\to 1-\varepsilon]\cup\gamma_{1,\varepsilon}\cup[1-\varepsilon\to z]} \frac{dx}{1-x}\left(\frac{dx}{x}\right)^{s-1},$$

which can be calculated using the coproduct formula of Theorem 0 applied associatively to the three paths along which the homotopy functional is being computed. In order to ensure that the conditions of this theorem are met, w must be bounded away from 1. In particular, this allows one to show that

$$\left|\int_{[w\to 1-\varepsilon]} \frac{dx}{x}\right| > \left|\int_{\gamma_{1,\varepsilon}^{-1}\cup[1-\varepsilon\to 0]\to z} \frac{dx}{x}\right|$$

for any $z\in[0,1]$ (with notation as in Theorem 0).

Now regarding the first two paths $[0\to 1-\varepsilon]$ and $\gamma_{1,\varepsilon}$, in fact the coproduct degenerates into the sum

$$\int_{[0\to 1-\varepsilon]} \frac{dx}{1-x}\left(\frac{dx}{x}\right)^{s-1} + \int_{\gamma_{1,\varepsilon}} \frac{dx}{1-x}\left(\frac{dx}{x}\right)^{s-1}$$

since $\int_{\gamma_{1,\varepsilon}} \frac{dx}{x} = 0$.

Hence, we obtain

$$\begin{aligned}
&\int_{[0\to 1-\varepsilon]\cup\gamma_{1,\varepsilon}\cup[1-\varepsilon\to w]} \frac{dx}{1-x}\left(\frac{dx}{x}\right)^{s-1}\\
= \ &\int_{[1-\varepsilon\to w]} \frac{dx}{1-x}\left(\frac{dx}{x}\right)^{s-1} + \sum_{n=0}^{\infty}\int_{[0\to 1-\varepsilon]\cup\gamma_{1,\varepsilon}} \frac{dx}{1-x}\left(\frac{dx}{x}\right)^{n}\cdot\int_{[1-\varepsilon\to w]}\left(\frac{dx}{x}\right)^{s-1-n}\\
= \ &\int_{[1-\varepsilon\to w]} \frac{dx}{1-x}\left(\frac{dx}{x}\right)^{s-1} + \sum_{n=0}^{\infty}\int_{[0\to 1-\varepsilon]} \frac{dx}{1-x}\left(\frac{dx}{x}\right)^{n}\cdot\int_{[1-\varepsilon\to w]}\left(\frac{dx}{x}\right)^{s-1-n}\\
&+\sum_{n=0}^{\infty}\int_{\gamma_{1,\varepsilon}} \frac{dx}{1-x}\left(\frac{dx}{x}\right)^{n}\cdot\int_{[1-\varepsilon\to w]}\left(\frac{dx}{x}\right)^{s-1-n}\\
= \ &Li_s(w) + \sum_{n=0}^{\infty}\int_{\gamma_{1,\varepsilon}} \frac{dx}{1-x}\left(\frac{dx}{x}\right)^{n}\cdot\int_{[1-\varepsilon\to w]}\frac{dx}{x}^{s-1-n},
\end{aligned}$$

by the comultiplication formula applied to (18). Now in the remaining sum, allowing ε to approach 0, the only non-zero integral about $\gamma_{1,\varepsilon}$ is the one for which $n=0$. This may be seen by writing the iterated integral as a contour integral, and recalling that here, $\operatorname{Re} s > 1$.

Consequently, we end up with

$$Li_s(w) + \lim_{\varepsilon\to 0}\int_{\gamma_{1,\varepsilon}} \frac{dx}{1-x}\cdot\int_{[1-\varepsilon\to w]}\frac{dx}{x}^{s-1} = Li_s(w) - 2\pi i\frac{(\log w)^{s-1}}{\Gamma(s)}.$$

□

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